Theorems
Complex analysis qualifying course
MSU, Spring 2017

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This document was made as a way to study the material from the spring semester complex analysis qualifying course at Michigan State University, in spring of 2017. It serves as a companion document to the "Definitions" review sheet for the same class. The textbook for the course was *Complex Function Theory*, by Donald Sarason, and these notes closely follow that text.

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1 Chapter 1: Complex Numbers

Proposition 1.1 (Basic Identities and Inequalities)**.** *Let* $z, z_1, z_2, z_3 \in \mathbb{C}$*. Let* $z = x + iy = r(\cos\theta + i\sin\theta)$ *and* $z_k = x_k + iy_k = r_k(\cos\theta_k + i\sin\theta_k)$*.*

$$
\begin{aligned}
|z| = |\overline{z}| &= \sqrt{z\overline{z}} \implies |z|^2 = z\overline{z} \\
\operatorname{Re} z &= \frac{1}{2}(z + \overline{z}) \\
\operatorname{Im} z &= \frac{1}{2i}(z - \overline{z}) \\
\overline{z_1 + z_2} &= \overline{z}_1 + \overline{z}_2 && \textit{(conjugation is a field automorphism)} \\
\overline{z_1 z_2} &= \overline{z}_1 \overline{z}_2 \\
|z_1 z_2| &= |z_1||z_2| \\
|z_1 + z_2| &\leq |z_1| + |z_2| && \textit{(triangle inequality)} \\
|z_1 + z_2 + \ldots + z_n| &\leq |z_1| + |z_2| + \ldots + |z_n| && \textit{(generalized triangle inequality)} \\
|z_1 - z_2| &\geq |z_1| - |z_2| \\
|z_1 + z_2|^2 + |\overline{z}_1 + \overline{z}_2|^2 &= 2\left(|z_1|^2 + |z_2|^2\right) && \textit{(parallelogram equality)} \\
z_1 z_2 &= r_1 r_2\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big) \\
\arg \overline{z} &= \arg(z^{-1}) = -\arg z \\
z^n &= r^n(\cos n\theta + i\sin n\theta) \quad \textit{for } n \in \mathbb{Z} && \textit{(De Moivre's Formula)} \\
z^{1/n} &= r^{1/n}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right) && \textit{for } n \in \mathbb{N} \textit{ and } k = 0, 1, \ldots, n-1
\end{aligned}
$$

Proposition 1.2. *Let* $z_1, z_2 \in \mathbb{C} \setminus \{0\}$*. Then they are positive multiples of each other if and only if* $z_1\overline{z}_2$ *is real and positive.*

Proposition 1.3. *Let* $p(x)$ *be a polynomial with real coefficients. If* $p(z) = 0$ *for some* $z \in \mathbb{C}$*, then* $p(\overline{z}) = 0$*. (That is, the Galois group of* $\mathbb{C}/\mathbb{R}$ *is just* $\mathbb{Z}/2\mathbb{Z}$*, the identity and complex conjugation.)*

Proposition 1.4. *Let* $z \in \mathbb{C} \setminus \{1\}$*. Then*

$$\sum_{k=0}^{n} z^k = 1 + z + z^2 + \ldots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

Proposition 1.5. *The sum of the nth roots of 1 equals zero for* $n \geq 2$*.*

2 Chapter 2: Complex Differentiation

Proposition 2.1 (Basic Properties of Complex Differentiation)**.** *Let* f *and* g *be complex-valued functions defined on an open set* G*.*

1. *If* f *is differentiable at* z_0*, then* f *is continuous at* z_0*.*

2. *(Leibniz rule) If f and g are differentiable at z_0, then $f+g$ and fg are differentiable at z_0, and $(f+g)'(z_0) = f'(z_0) + g'(z_0)$ and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.*

3. *If f and g are differentiable at z_0 and $g(z_0) \neq 0$, then f/g is differentiable at z_0. (There is a quotient rule, but just apply the product rule to $f\frac{1}{g}$.)*

4. *(Chain rule) If f is differentiable at z_0 and g is differentiable at $f(z_0)$, then the composition $g \circ f$ is differentiable at z_0 and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.*

Proposition 2.2. *Polynomial functions are holomorphic.*

Proposition 2.3. *Rational functions are holomorphic everywhere that their denominator is nonzero.*

Proposition 2.4 (Cauchy-Riemann Equations)**.** *Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be a complex function defined on an open set G containing z_0. Then f is differentiable at z_0 if and only if u, v are differentiable at z_0 and*

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

In that case,

$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0)$$

Proposition 2.5. *In polar form, the Cauchy-Riemann equations are*

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proposition 2.6. *Let the complex-valued function $f = u + iv$ be defined in the open subset $G \subset \mathbb{C}$, and assume that u and v have first partial derivatives in G. Then f is differentiable at each point where those partial derivatives are continuous and satisfy the Cauchy-Riemann equations.*

Proposition 2.7. *Let f be holomorphic in an open disk D. If any of the following hold for all $z \in D$, then f is constant in D: $f'(z) = 0$, $f(z) \in \mathbb{R}$, $|f| = c$, $\arg f = c$.*

Proposition 2.8. *Let f be holomorphic on the open set G. Then $\overline{f(\overline{z})}$ is holomorphic on $\{\overline{z} : z \in G\}$.*

Proposition 2.9. *A function f is differentiable at z_0 if and only if $\frac{\partial}{\partial \overline{z}}(z_0) = 0$.*

Proposition 2.10. *If f is differentiable at z_0, then $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.*

Proposition 2.11. *Let f be holomorphic on an open set G and let $\gamma : I \to G$ be a curve such that γ is differentiable at t_0, and let $z_0 = \gamma(t_0)$. Then the curve $f \circ \gamma : I \to f(G)$ is differentiable at t_0 and*

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0)$$

Proposition 2.12 (Holomorphic Maps are Conformal where Derivative is Nonzero)**.** *Let f be holomorphic on an open set G and let $z_0 \in G$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be curves such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, and γ_j is regular at t_j. Then the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ is equal to the angle between γ_1 and γ_2.*

Proposition 2.13 (Conformal implies Holomorphic)**.** *Let $f : G \to \mathbb{C}$ where $G \subset \mathbb{C}$ is open, and suppose that $\operatorname{Re} f$ and $\operatorname{Im} f$ have continuous first partial derivatives. If f is conformal at each $z_0 \in G$, then f is holomorphic and $f' \neq 0$ in G.*

Proposition 2.14. *Let $f : G \to \mathbb{C}$. Then f is holomorphic if and only if its real and imaginary parts are harmonic.*

Proposition 2.15. *Holomorphic functions are harmonic.*

Proposition 2.16. *Let $u, v : G \to \mathbb{C}$ where $G \subset \mathbb{C}$ is open and suppose u, v are of class C^2. Then u, v are harmonic conjugates if and only if $u + iv$ is holomorphic.*

3 Chapter 3: Linear Fractional Transformations

Proposition 3.1. *The map $\mathbb{CP}^1 \to \overline{\mathbb{C}}$ given by $[z_1, z_2] \mapsto \frac{z_1}{z_2}$ is well-defined, and is a bijection. (The RHS is taken to be ∞ when $z_2 = 0$.)*

Proposition 3.2. *A linear fractional transformation gives a bijection $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.*

Proposition 3.3. *Linear fractional transformations form a group under function composition. That is, LFTs are closed under composition and the inverse of a linear fractional transformation is a linear fractional transformation.*

Proposition 3.4. *Let ϕ_1, ϕ_2 be linear fractional transformations induced by matrices M_1, M_2 respectively. Then $\phi_1\phi_2$ is induced by M_1M_2. If ϕ is a linear fractional transformation induced by M, then ϕ^{-1} is induced by M^{-1}. That is, the map*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \left(z \mapsto \frac{az + b}{cz + d} \right)$$

from $\mathrm{GL}(2, \mathbb{C})$ to the group of LFTs is a group homomorphism. The kernel is

$$H = \{\lambda I_2 : \lambda \in \mathbb{C} \setminus \{0\}\}$$

where I_2 is the identity of $\mathrm{GL}(2, \mathbb{C})$. Thus the LFT group is isomorphic to

$$\mathrm{GL}(2, \mathbb{C})/H$$

Proposition 3.5. *Every linear fractional transformation can be written as a product of a dilation, rotation, translation, and inversion. That is, the dilations, rotations, translations, and the inversion map generate the group of linear fractional transformations.*

Proposition 3.6. *Linear fractional transformations are conformal. Consequently, they are holomorphic and have a first derivative that never vanishes.*

The next lemma seems very out of place, since it is purely topological. However, it is very useful when dealing with linear fractional transformations.

Lemma 3.7. *Let X be a topological space, and $A \subset X$. Let $\phi : X \to X$ be a homeomorphism so that $\phi(A) = A$ and $\phi|_A : A \to A$ is a homeomorphism. Then $\phi(\partial A) = \partial A$ and $\phi|_{\partial A} : \partial A \to \partial A$ is a homeomorphism.*

Corollary 3.8. *Let ϕ be a linear fractional transformation, and let C be a clircle dividing $\mathbb{C}$ into two disconnected regions X, Y. Since $\phi(C)$ is a clircle, it divides $\mathbb{C}$ into two disconnected regions $\widetilde{X}, \widetilde{Y}$. Then $\phi|_X$ is a bijection $X \to \widetilde{X}$ or a bijection $X \to \widetilde{Y}$.*

Intuitively speaking, the above result says that a linear fractional transformation that maps a given clirlce to another clircle must map one "side" of the clircle in the domain to one "side" of the image clircle. (Where "side" refers to the inside/outside if it is a circle, and "side" refers to top/bottom or right/left if it is a line.)

Proposition 3.9. *A linear fractional transformation has exactly one or two fixed points.*

Proposition 3.10. *Let z_1, z_2, z_3 be distinct poins in $\overline{\mathbb{C}}$, and w_1, w_2, w_3 be distinct points in $\overline{\mathbb{C}}$. There is a unique linear fractional transformation ϕ so that $\phi(z_i) = w_i$.*

Proposition 3.11. *Let z_1, z_2, z_3, z_4 be distinct points in $\overline{\mathbb{C}}$ and ϕ a linear fractional transformation. Then*

$$(\phi z_1, \phi z_2; \phi z_3, \phi z_4) = (z_1, z_2; z_3, z_4)$$

Proposition 3.12. *All translations except for translation by zero are mutually conjugate.*

Proposition 3.13. *Let f be a fractional linear transformation with a unique fixed point at ∞. Then f is a translation.*

Proposition 3.14. *A linear fractional transformation with exactly one fixed point is conjugate to a translation.*

Proposition 3.15. *The image of a clircle under a linear fractional transformation is a clircle.*

Proposition 3.16. *Let z_1, z_2, z_3, z_4 be distinct points in $\overline{\mathbb{C}}$. They all lie on a clircle if and only if the cross ratio $(z_1, z_2; z_3, z_4)$ is real.*

Proposition 3.17. *A clircle is uniquely determined by three points.*

4 Chapter 4: Elementary Functions

Proposition 4.1. *Let $z_1, z_2 \in \mathbb{C}$. Then $e^{z_1} e^{z_2} = e^{z_1 + z_2}$.*

Proposition 4.2. *The complex exponential is holomorphic. Its derivative is itself.*

Proposition 4.3. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $f' = f$, then f is a constant multiple of e^z.*

Proposition 4.4. *All complex trigonometric and hyperbolic functions are holomorphic everywhere they are defined.*

Proposition 4.5. *Let $z \in \mathbb{C}$. Then*

$$\cos z = \cosh iz \qquad \sin z = -i \sinh iz$$

Proposition 4.6. *Let $z \in \mathbb{C} \setminus \{0\}$. If w is a logarithm of z, then $w = \ln|z| + i \arg z$.*

Proposition 4.7. *If f is holomorphic on a disk containing z_0 and $f(z_0) \neq 0$, then there is a branch of $\log f$ on a disk containing z_0.*

Proposition 4.8. *Let G be an open connected subset of $\mathbb{C} \setminus \{0\}$. There exists a branch of* $\arg$ *in G if and only if there exists a branch of* $\log$ *in G.*

Proposition 4.9. *If α is a branch of $\arg z$ in G, then $\alpha + 2\pi n$ is another branch for any $n \in \mathbb{Z}$. Conversely, if α_1, α_2 are branches of* $\arg$ *in G, then they differ by an integer mulitple of 2π.*

Proposition 4.10. *If ℓ is a branch of $\log z$ in G, then $\ell + 2\pi i n$ is another branch for any $n \in \mathbb{Z}$. Conversely, any two branches of ℓ in G differ by an integer multiple of $2\pi i$.*

Proposition 4.11. *Let ℓ be a branch of $\log z$ in the open connected set G. Then ℓ is holomorphic and $\ell'(z) = \frac{1}{z}$.*

Proposition 4.12. *If there is a branch of $\log f$ in G, then it is a holomorphic function and its derivative is $\frac{f'}{f}$.*

Proposition 4.13. *If h is a branch of $f^{1/n}$, then h is holomorphic and $\frac{h'}{h} = \frac{f'}{nf}$.*

5 Chapter 5: Power Series

Proposition 5.1. *If $\sum_{n=0}^{\infty} c_n$ converges then $\lim_{n\to\infty} c_n = 0$.*

Proposition 5.2. *Let $z \in \mathbb{C}$. The geometric series $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$ if $|z| < 1$ and diverges for $|z| \geq 1$.*

Proposition 5.3. *If $\sum_{n=0}^{\infty} c_n$ converges, then*

$$\left|\sum_{n=0}^{\infty} c_n\right| \leq \sum_{n=0}^{\infty} |c_n|$$

Proposition 5.4. *If a series converges absolutely, then it converges.*

Proposition 5.5. *A series of functions converges uniformly on S if and only if it is uniformly Cauchy on S.*

Proposition 5.6. *Let g_n be a sequence of complex valued functions that converges uniformly on G. Then g_n converges uniformly on any subset of G.*

Proposition 5.7. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a power series. The region of convergence is either* $\{z_0\}$, $\mathbb{C}$, *or an open disk* $|z-z_0|<R$, *possibly including points on the boundary of that disk. If it converges on the disk* $|z-z_0|<R$, *then it converges absolutely and locally uniformly.*

Proposition 5.8. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *and* $\sum_{n=0}^{\infty} b_n$ *be power series with respective radii of convergence* R_1, R_2. *If there exists* $M>0$ *so that* $|a_n| \le M|b_n|$ *for all but finitely many* n, *then* $R_1 \le R_2$.

Proposition 5.9. *Let* a_n *be a sequence of real numbers. Then* $\lim a_n$ *exists if and only if* $\limsup a_n = \liminf a_n$. *If it exists, then* $\lim a_n = \limsup a_n = \liminf a_n$.

Proposition 5.10. *Let* a_n, b_n *be real sequences. Then*

$$\limsup(a_n+b_n) \le \limsup a_n + \limsup b_n$$

provided the sum on the right is well-defined (that is, it isn't $\infty-\infty$*). If either sequence converges, then we get equality.*

Proposition 5.11. *Let* a_n, b_n *be positive real sequences. Then*

$$\limsup(a_n b_n) \le (\limsup a_n)(\limsup b_n)$$

as long as the product on the right is meaningful (i.e. not $0\cdot\infty$*). If either sequence converges, we get equality.*

Proposition 5.12 (Cauchy-Hadamard Theorem)**.** *The radius of convergence of* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *is*

$$\frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

Proposition 5.13. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *and* $\sum_{n=0}^{\infty} b_n$ *be power series with respective radii of convergence* R_1, R_2. *The radius of convergence of* $\sum_{n=0}^{\infty}(a_n+b_n)z^n$ *is at least* $\min(R_1, R_2)$. *The radius of convergence of* $\sum_{n=0}^{\infty} a_n b_n z^n$ *is at least* R_1R_2 *(as long as* R_1R_2 *isn't* $0\cdot\infty$*).*

Proposition 5.14. *A power series* $\sum_{n=0}^{\infty} a_n z^n$ *and its termwise derivative* $\sum_{n=1}^{\infty} n a_n z^{n-1}$ *have the same radius of convergence.*

Proposition 5.15 (Ratio Test)**.** *If*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

exists, then the above limit is the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Proposition 5.16. *Suppose that the power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has positive radius of convergence* R. *Then the function* f *represented by the above power series in the disk* $|z-z_0|<R$ *is holomorphic, and* f' *is represented in the same disk by the termwise derivative* $\sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1}$.

Proposition 5.17. *A function represented by a power series in a disk* $|z-z_0|<R$ *is infinitely differentiable on that disk.*

Proposition 5.18. *The power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has infinite radius of convergence, and represents the function e^z *on all of* $\mathbb{C}$.

Proposition 5.19. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *and* $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ *be power series with the same center, both with positive radii of convergence* R_1, R_2 *respectively. Then their Cauchy product converges in the disk* $|z-z_0| < \min(R_1, R_2)$. *The function represented by the Cauchy product is the product of functions represented by the original series. That is,*

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z-z_0)^n = \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n \right) \left(\sum_{n=0}^{\infty} b_n (z-z_0)^n \right)$$

6 Chapter 6: Complex Integration

Proposition 6.1 (Linearity of Complex Integral). *Let* $\phi_1, \phi_2 : [a,b] \to \mathbb{C}$ *be piecewise continuous and let* $c_1, c_2 \in \mathbb{C}$. *Then*

$$\int_a^b c_1\phi_1(t) + c_2\phi_2(t)dt = c_1 \int_a^b \phi_1(t)dt + c_2 \int_a^b \phi_2(t)dt$$

Proposition 6.2 (Fundamental Theorem of Calculus). *Let* $\phi : [a,b] \to \mathbb{C}$ *be piecewise* C^1. *Then*

$$\int_a^b \phi'(t)dt = \phi(b) - \phi(a)$$

Proposition 6.3. *Let* $\phi : [a,b] \to \mathbb{C}$ *be piecewise continuous. Then*

$$\left| \int_a^b \phi(t)dt \right| \leq \int_a^b |\phi(t)|dt$$

Proposition 6.4 (Linearity of Complex Line Integral). *Let* $G_1, G_2 \subset \mathbb{C}$ *and* $f_1 : G_1 \to \mathbb{C}$ *and* $f_2 : G_2 \to \mathbb{C}$. *Let* $\gamma : [a,b] \to G_1 \cap G_2$ *be piecewise* C^1, *and* $c_1, c_2 \in \mathbb{C}$. *Then*

$$\int_\gamma c_1 f_1(z) + c_2 f_2(z)dz = c_1 \int_\gamma f_1(z)dz + c_2 \int_\gamma f_2(z)dz$$

Proposition 6.5 (Partitioning of Curves of a Line Integral). *Let* $G \subset \mathbb{C}$ *and* $f : G \to \mathbb{C}$. *Let* $\gamma : [a,c] \to G$ *be piecewise* C^1, *and let* $b \in [a,c]$. *Define* $\gamma_1 = \gamma|_{[a,b]}$ *and* $\gamma_2 = \gamma|_{[b,c]}$. *Then*

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

Proposition 6.6. *Let* $G \subset \mathbb{C}$ *be open, and let* $f : G \to \mathbb{C}$ *be holomorphic, and assume that* f' *is continuous. Let* $\gamma : [a,b] \to G$ *be a piecewise* C^1 *curve. Then*

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

In particular, if γ *is a closed curve, then the above integral is zero.*

Proposition 6.7. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = z_0 + re^{it}$. (Note that γ parametrizes the circle $|z - z_0| = r$ traversed once clockwise.) Then*

$$\int_\gamma \frac{1}{(z - z_0)^n} dz = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Proposition 6.8. *Let $G \subset \mathbb{C}$ be open and $f : G \to \mathbb{C}$ and let $\gamma : [a, b] \to G$ be piecewise C^1. Let $\gamma_1 = \gamma \circ \beta$ be a reparametrization of γ. Then*

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz$$

Because of this equality, when speaking of an integral of a function over a curve, one is free to choose a convenient parametrization to compute the integral. (Note: Reversing the direction of a curve is NOT a reparametrization.)

Proposition 6.9. *Reversing the direction of a curve changes the sign of the integral over that curve. That is,*

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$$

Proposition 6.10. *Let $G \subset \mathbb{C}$ and $f : G \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to G$ be piecewise C^1. Let M be the maximum of $|f|$ on γ, that is,*

$$M = \max\{|f(\gamma(t))| : t \in [a, b]\}$$

Note that M exists by the extreme value theorem becase the trace of γ is compact. Then

$$\left| \int_\gamma f(z)dz \right| \leq ML(\gamma)$$

where $L(\gamma)$ is the length of γ.

Proposition 6.11 (Passing a Limit through an Integral)**.** *Let $G \subset \mathbb{C}$ and let $f_n : G \to \mathbb{C}$ be a sequence of continuous functions. Let $\gamma : [a, b] \to G$, and suppose that f_n converges uniformly to f on $\gamma([a, b])$. Then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z) = \int_\gamma f(z)dz$$

WARNING: Uniform convergence is necessary!

7 Chapter 7: Core Versions of Cauchy's Theorem

Proposition 7.1 (Cauchy's Theorem for Triangles)**.** *Let $G \subset \mathbb{C}$ be open, and let $f : G \to \mathbb{C}$ be holomorphic. Let T be a triangle in $\mathbb{C}$ such that T and its interior are contained in G. Then*

$$\int_T f(z)dz = 0$$

Note that this is subsumed by later, more general, versions of Cauchy's Theorem.

Proposition 7.2 (Cauchy's Theorem for a Star-Shaped Region)**.** *Let $G \subset \mathbb{C}$ be star shaped and open. Let $f : G \to \mathbb{C}$ be holomorphic. Let $\gamma : [a, b] \to G$ be a piecewise C^1 curve. Then*

$$\int_\gamma f(z)dz = 0$$

Note that this is subsumed by later, more general, versions of Cauchy's Theorem.

Proposition 7.3. *Let $G \subset \mathbb{C}$ be open and star shaped, and let $f : G \to \mathbb{C}$ be holomorphic. Then f has a primitive in G. (Later, we will only need G to be simply connected.)*

Proposition 7.4 (Cauchy's Formula for a Circle)**.** *Let C be a counterclockwise oriented circle and let f be holomorphic on an open set containing C and its interior. Then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

for z in the interior of C. (That is, we can recover the value of f at the center of a circle from the values on the circle.)

Proposition 7.5 (Mean Value Property)**.** *Let f be holomorphic in the disk $|z - z_0| < R$. Then for $0 < r < R$,*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt$$

That is, the value of f at the center of the circle $|z - z_0| = r$ is the average of the values along the circle.

Proposition 7.6 (Mean Value Property, 2)**.** *Let f be holomorphic in the disk $|z - z_0| < R$. Then for $0 < r < R$,*

$$f(z_0) = \frac{1}{\pi r^2} \iint_{|z - z_0| < r} f(z)\, dA$$

That is, the value of f at the center of the disk $|z - z_0| < r$ is equal to the average of the values of f on that disk.

Proposition 7.7 (Holomorphic functions have local power series representations)**.** *Let f be be holomorphic on an open set containing the disk $|z - z_0| < r$. Then there is a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ representing f in that disk. In particular, f is represented by its Taylor series:*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

on $|z - z_0| < R$.

Proposition 7.8 (Cauchy Integral Formula)**.** *The derivative of a holomorphic function is holomorphic. In particular, if f is holomorphic on G, then for $|z - z_0| < r < \operatorname{dist}(z_0, G^c)$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(w)}{(w - z)^{n+1}} dw$$

Consequently, if f has the local power series representation $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, then

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

Proposition 7.9 (Converse of Goursat's Lemma, sometimes called Morera's Theorem)**.** *Let f be a continuous complex valued function on an open subset G of $\mathbb{C}$. If $\int_T f(z)dz = 0$ for every triangle T with interior contained in G, then f is holomorphic.*

Proposition 7.10 (Morera's Theorem for Rectangles)**.** *Let f be a continuous complex valued function on an open subset G of $\mathbb{C}$. If $\int_R f(z)dz = 0$ for every rectangle R with interior contained in G, then f is holomorphic.*

Proposition 7.11 (Liouville's Theorem)**.** *If f is entire and bounded, then it is constant.*

Proposition 7.12 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with complex coefficients can be factored over $\mathbb{C}$ into linear factors.*

Proposition 7.13. *Let $f : G \to \mathbb{C}$ be holomorphic, and let $z_0 \in G$ be a zero of order m. Then the Taylor series of f centered at z_0 is*

$$\sum_{n=m}^{\infty} a_n(z - z_0)^n$$

Proposition 7.14. *Zeroes of finite order of holomorphic functions are isolated. That is, if z_0 is a zero of order m of a holomorphic function f, then there exists $r > 0$ so that $f(z) \neq 0$ for $z \in B(z_0, r)$.*

Proposition 7.15. *If $f : G \to \mathbb{C}$ has a zero of infinite order at z_0, then f is the zero function on the connected component of G containing z_0.*

Proposition 7.16. *Let $f : G \to \mathbb{C}$ be holomorphic with G connected, with f not the zero function. Then each zero of f is of finite order, and $f^{-1}(0)$ has no limit points in G.*

Proposition 7.17. *Let $f : G \to \mathbb{C}$ be holomorphic and have a zero of order m at z_0. Then there is a branch of $f^{\frac{1}{m}}$ in a disk centered at z_0.*

Proposition 7.18 (Identity Principle)**.** *Let $f, g : G \to \mathbb{C}$ be holomorphic with G connected. If $f(z) = g(z)$ for all z in a subset of G that has a limit point in G, then $f = g$.*

Proposition 7.19 (Weierstrass Convergence Theorem)**.** *Let $G \subset \mathbb{C}$ be open and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of holomorphic functions in G that converges locally uniformly in G to the function f. Then f is holomorphic, and for each $n \in \mathbb{N}$, the sequence $\{f_k^{(n)}\}_{k=1}^{\infty}$ converges locally uniformly to $f^{(n)}$. That is, the locally uniform limit of holomorphic functions is holomorphic.*

Proposition 7.20 (Maximum Modulus Principal)**.** *Let f be a nonconstant holomorphic function in the open connected set $G \subset \mathbb{C}$. Then $|f|$ does not attain a local maximum in G. As a consequence, if $K \subset G$ is compact, then $|f|$ attains its maximum over K only at points on the boundary of K.*

Proposition 7.21. *Let f be a nonconstant holomorphic function in the connected open subset $G \subset \mathbb{C}$. Then $|f|$ can attain a local minimum only at a zero of f.*

Proposition 7.22 (Schwarz's Lemma)**.** *Let f be a holomorphic map of the open unit disk to itself so that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all z in the disk. The inquality is strict at all points except 0, unless f is a rotation, i.e. $f(z) = \lambda z$ where $|\lambda| = 1$.*

Proposition 7.23 (Existence of a Harmonic Conjugate on a Convex Set)**.** *Let u be a real-valued harmonic function in the convex open subset $G \subset \mathbb{C}$. Then there is a holomorphic function $g : G \to \mathbb{C}$ so that $u = \operatorname{Re} g$. The function g is unique up to addition of an imaginary constant. (That is, $\operatorname{Im} g$ is a harmonic conjugate to u.)*

Proposition 7.24. *Harmonic functions are infinitely differentiable.*

Proposition 7.25 (Mean Value Property for Harmonic Functions)**.** *Let $u : G \to \mathbb{C}$ be harmonic with $G \subset \mathbb{C}$ open. Let $z_0 \in G$. Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt$$

for $0 < r < \operatorname{dist}(z_0, G^c)$.

Proposition 7.26 (Identity Principle for Harmonic Functions)**.** *Let $G \subset \mathbb{C}$ be open and connected. Let $u, v : G \to \mathbb{C}$ be harmonic functions that agree on a nonempty open subset of G. Then $u = v$.*

Proposition 7.27 (Maximum Modulus Priniple for Harmonic Functions)**.** *Let $G \subset \mathbb{C}$ be open an connected. Let $u : G \to \mathbb{R}$ be a nonconstant harmonic function. Then u does not attain a local maximum in G.*

8 Laurent Series and Isolated Singularities

Note: Prof Schenker presented this material in a different order in class, giving a different definition for singularities, but it all turns out to be logically equivalent.

Proposition 8.1 (Generalized Cauchy-Hadamard Theorem)**.** *Consider the Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and let*

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{1/n} \qquad R_2 = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

If $R_1 < R_2$, then the Laurent series converges absolutely and locally uniformly in the annulus $R_1 < |z - z_0| < R_2$.

Proposition 8.2. *Consider the Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and let*

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{1/n} \qquad R_2 = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

For z such that $R_1 < |z - z_0| < R_2$ define $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. Then f is holomorphic on the annulus. In addition, for r satsifying $R_1 < r < R_2$, let C_r denote the circle $|z - z_0| = r$, then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Proposition 8.3 (Cauchy's Theorem for Concentric circles). *Let f be holomorphic in the annulus $R_1 < |z - z_0| < R_2$, and for $R_1 < r < R_2$ let C_r be the circle $|z - z_0| = r$ oriented counterclockwise. Then $\int_{C_r} f(z)dz$ is independent of r (for $R_1 < r < R_2$.)*

Proposition 8.4 (Cauchy's Formula for an Annulus). *Let f be holomorphic in the annulus $R_1 < |z - z_0| < R_2$, and for $R_1 < r < R_2$ let C_r be the circle $|z - z_0| = r$ oriented counterclockwise. If $R_1 < r_1 < |w - w_0| < r_2 < R_2$, then*

$$f(w) = \frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(z)}{z-w}dz - \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(z)}{z-w}dz$$

Proposition 8.5. *Let f be holomorphic in the annulus $R_1 < |z - z_0| < R_2$. Then f has a Laurent series representation on that annulus.*

Proposition 8.6 (Criterion for a Removable Singularity). *Let f be holomorphic with an isolated singularity at z_0. Then f is bounded in some punctured disk with center z_0 if and only if z_0 is a removable singularity.*

Proposition 8.7 (Criterion for a Pole). *Let f be holomorphic with an isolated singularity at z_0. If*

$$\lim_{z\to z_0} |f(z)| = \infty$$

then z_0 is a pole of f.

Proposition 8.8 (Casorati-Weierstrass Theorem). *Let $f : G \to \mathbb{C}$ be holomorphic with an essential isolated singularity at z_0. Then for any $w \in \mathbb{C}$, there is a sequence $(z_n)_{n=1}^{\infty}$ in G so that*

$$\lim_{n\to\infty} z_n = z_0 \qquad \lim_{n\to\infty} f(z_n) = w$$

The above proposition says that functions behave very badly near essential singularities. It says that not only does not limit as $z \to z_0$ of $f(z)$ not exist, it actually can take ANY value in $\mathbb{C}$, for a suitably chosen path.

The following result was not proven in our class, but is included in the book for interest's sake.

Proposition 8.9 (Picard's Theorem). *Let f be holomorphic with an essential isolated singularity at z_0. Then in any punctured disk centered at z_0, the range of f includes every complex value infinitely many times, with possibly one exception.*

Proposition 8.10. *Let f, g be holomorphic on an open set containing z_0 and suppose g has a simple zero at z_0. Then $\mathrm{res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$.*

Proposition 8.11 (Computing a Residue at a Pole). *Let f be holomorphic on an open set containing z_0 and let f have a pole of order k at z_0. Then*

$$\mathrm{res}_{z_0} f = \frac{1}{(k-1)!}\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]$$

In particular, if z_0 is a simple pole, $(k = 1)$, then

$$\mathrm{res}_{z_0} f = \lim_{z\to z_0} (z - z_0)f(z)$$

Proposition 8.12 (Baby Residue Theorem)**.** *Let f be holomorphic with an isolated singularity at z_0. Let C be a counterclockwise oriented circle centered at z_0 so that f is holomorphic on the punctured interior of C. Then*

$$\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f$$

We can rewrite this as

$$\operatorname{res} z_0 f = \frac{1}{2\pi i}\int_C f(z)dz$$

9 Cauchy's Theorem

Proposition 9.1. *Let $\phi : [a,b] \to \mathbb{C} \setminus \{0\}$ be continuous. Then there is a continuous $\psi : [a,b] \to \mathbb{C} \setminus \{0\}$ so that $\phi = e^{\psi}$. Furthermore, ψ is unique up to addition integer multiples of $2\pi i$.*

Proposition 9.2. *Let $\phi : [a,b] \to \mathbb{C} \setminus \{0\}$ be piecewise C^1. Let c be a value of $\log \phi(a)$. Then define $\psi : [a,b] \to \mathbb{C} \setminus \{0\}$ by*

$$\psi(t) = c + \int_a^t \frac{\phi'(s)}{\phi(s)}dx$$

Then ψ is continuous and $\phi = e^{\psi}$.

Proposition 9.3. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 curve and f holomorphic on an open set containing γ and nonvanishing on γ. Then*

$$\Delta(\log f, \gamma) = \int_\gamma \frac{f'(z)}{f(z)}dz$$

Proposition 9.4. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 closed curve, and let z_0 be a point not in the trace of γ. Then*

$$\operatorname{ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}dz$$

Since the RHS is a continuous function of z_0, and integer valued, it is constant on each connected component of $\mathbb{C} \setminus \gamma([a,b])$. In particular, the index must be zero on the unbounded component.

Proposition 9.5. *Let $\Gamma = \sum_j n_j \gamma_j$ be a contour. Then*

$$\operatorname{ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z - z_0}dz$$

Proposition 9.6 (The Separation Lemma)**.** *Let $G \subset \mathbb{C}$ be open, and let $K \subset G$ be compact. Then there is a simple contour Γ in $G \setminus K$ such that $K \subset \operatorname{int}\Gamma \subset G$, and such that if f is holomorphic in G, then*

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz$$

Proposition 9.7 (Cauchy's Theorem)**.** *Let $G \subset \mathbb{C}$ be open and let Γ be a contour with interior contained in G, and let $f : G \to \mathbb{C}$ be holomorphic. Then*

$$\int_\Gamma f(z)dz = 0$$

(Keep in mind that Γ doesn't wind around anything outside G.)

Proposition 9.8. *Let $\phi : [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous. Then there is a continuous $\psi : [0,1] \times [0,1] \to \mathbb{C}$ such that $\phi = e^\psi$. The function ψ is unique up to addition of integer multiples of $2\pi i$.*

Proposition 9.9 (Homotopic Loops give Same Winding Number)**.** *Let $G \subset \mathbb{C}$ be open. Let γ_0, γ_1 be closed piecewise C^1 curves in G that are homotopic in G. Then $\operatorname{ind}_{\gamma_0}(z) = \operatorname{ind}_{\gamma_1}(z)$ for $z \in \mathbb{C} \setminus G$. (Note that it is important that $z \notin G$.)*

Proposition 9.10 (Homotopy Version of Cauchy's Theorem)**.** *Let $G \subset \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be holomorphic. Let γ_0, γ_1 be closed piecewise C^1 curves in G that are homotopic in G. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

10 Residue Theorem and Riemann Mapping Theorem

Proposition 10.1. *Every star shaped domain is simply connected.*

Proposition 10.2 (Winding Number Criterion)**.** *Let $G \subset \mathbb{C}$ be a domain. Then G is simply connected if and only if every contour Γ in G has winding number zero around every point in $\mathbb{C} \setminus G$.*

Proposition 10.3 (Cauchy's Theorem for Simply Connected Domains)**.** *Let $G \subset \mathbb{C}$ be a simply connected domain, and let $f : G \to \mathbb{C}$ be holomorphic, and let Γ be a contour in G. Then*

$$\int_\Gamma f(z)dz = 0$$

Note that this is a special case of the Residue Theorem.

Proposition 10.4 (Existence of Primitive Criterion)**.** *Let $G \subset \mathbb{C}$ be a domain. Then G is simply connected if and only if every holomorphic function $f : G \to \mathbb{C}$ has a primitive.*

Proposition 10.5 (Existence of Logarithms)**.** *Let $G \subset C$ be a simply connected domain, and let $f : G \to \mathbb{C} \setminus \{0\}$ be holomorphic. Then there is a branch of $\log f$ in G.*

Proposition 10.6 (Existence of Harmonic Conjugates)**.** *Let $G \subset \mathbb{C}$ be a simply connected domain, and let $u : G \to \mathbb{R}$ be harmonic. Then u has a harmonic conjugate in G, which is unique up to an additive constant.*

Proposition 10.7 (Partial Equivalence of Definitions of Simply Connected)**.** *Let $G \subset \mathbb{C}$ be a domain. If every closed curve in G is nullhomotopic, then G is simply connected. (The converse is also true, but proven later, using the Riemann Mapping Theorem.)*

Proposition 10.8 (Residue Theorem)**.** *Let $G \subset \mathbb{C}$ be a domain, and let Γ be a contour with interior contained in G. Let f be holomorphic in G except for isolated singularities $z_1, \ldots, z_p$, none of which lies on Γ. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{p} \operatorname{ind}_\Gamma(z_k) \operatorname{res}_{z_k}(f)$$

Proposition 10.9 (Cauchy Integral Formula)**.** *Let $G \subset \mathbb{C}$ be a domain, and let $f : G \to \mathbb{C}$ be holomorphic, and let Γ be a simple contour with interior in G. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz$$

for $z_0 \in \operatorname{int} \Gamma$. More generally,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for $z_0 \in \operatorname{int} \Gamma$.

Proposition 10.10 (Argument Principle)**.** *Let $G \subset \mathbb{C}$ be a domain and let Γ be a simple contour with interior contained in G. Let $f : G \to \mathbb{C}$ be holomorphic and nonvanishing on the trace of Γ. Then the number of zeroes of f in the interior of Γ (counting multiplicities) is*

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz$$

Proposition 10.11 (Rouche's Theorem)**.** *Let $G \subset \mathbb{C}$ be a domain, and let $K \subset G$ be compact. Let $f, g : G \to \mathbb{C}$ be holomorphic such that*

$$|f(z) - g(z)| < |f(z)| \qquad \forall z \in \partial K$$

Then f, g have the same number of zeroes in the interior of K (counting multiplicities).

Proposition 10.12 (Hurwtiz's Theorem)**.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of holomorphic functions on a domain G converging locally uniformly in G to a nonconstant function f. If f has at least m zeroes in G, then all but finitely many f_n have at least m zeroes in G. Consequently, if infinitely many f_n are univalent (injective), then f is univalent.*

Proposition 10.13 (Local Mapping Theorem)**.** *Let f be a nonconstant holomorphic function in the domain G. Let $z_0 \in G$ and let $w_0 = f(z_0)$. Let m be the order of the zero of $f - w_0$ at z_0. For every sufficiently small $\delta > 0$, there exists $\epsilon > 0$ such that every value w satisfying $0 < |w - w_0| < \epsilon$ is assumed by f at exactly m distinct points in the punctured disk $0 < |z - z_0| < \delta$, with multiplicity 1 at each of those points.*

Proposition 10.14 (Open Mapping Theorem)**.** *Let G be a domain, and let $f : G \to \mathbb{C}$ be holomorphic. Then f is an open map.*

Proposition 10.15 (Local Inverses)**.** *If f is holomorphic and $f'(z_0) \neq 0$, then there is a disk centered at z_0 on which f is univalent.*

Proposition 10.16. *A univalent holomorphic function has a nowhere vanishing derivative.*

Proposition 10.17. *The inverse of a univalent holomorphic function is holomorphic.*

Proposition 10.18 (Stieltjes-Osgood Theorem)**.** *A locally uniformly bounded sequence of holomorphic functions (on a domain in $\mathbb{C}$) has a locally uniformly convergent subsequence. (That is, it is a normal family.)*

Proposition 10.19 (Arzela-Ascoli Theorem)**.** *Let X be a compact metric space, and let $C(X)$ be the space of continuous functions $f : X \to \mathbb{R}$. Then $F \subset C(X)$ is equicontinuous and pointwise bounded if and only if it is a normal family.*

Proposition 10.20. *If a domain is conformally equivalent to a simply connected domain, then it is simply connected.*

Proposition 10.21 (Riemann Mapping Theorem)**.** *Every simply connected domain in $\mathbb{C}$ except for $\mathbb{C}$ itself is conformally equivalent to the unit disk. (That is to say, if $G \subset \mathbb{C}$ is not equal to $\mathbb{C}$, then there is a univalent holomorphic function $f : D \to G$, where D is the open unit disk.)*